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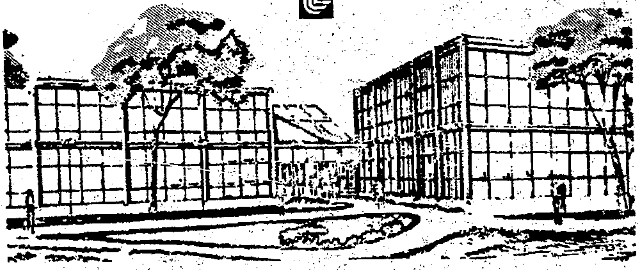

Single Particle Behavior in Plasmas

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Introduction

One cannot expect to memorize the content of such an intense course as this and considerable emphasis has been placed on collecting and cataloging key results and principal references. The behavior of single particles in a plasma is basic to the rest of the course, but not elementary. The subject is well covered in many text books and the purpose of this paper is merely to collect, in a brief form, the essential formulae and mathematical methods.

The paper follows the history of a neutral atom or molecule into a plasma - ionization, dissociation, radiation, - until it becomes a set of charged particles moving in the electromagnetic fields of the plasma system. The various useful forms of the method of averaging are displayed and applied to calculation of constants of motion. The breakdown of these constants is discussed along with some of the implications for fusion systems.

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3. Motion of Charged Particles in Electromagnetic Fields

Charged particle motions are generally complicated and, in designing fusion devices, one tries to simplify the motions by use of symmetries or constants of the particle motions to provide confinement within the device. The equations of motion in an electric field $\vec{E}(\vec{x},t)$ and magnetic field $\vec{B}(X,t)$ are, in Gaussian units,

$$\frac{d\vec{X}}{dt} = \vec{V} \quad , \quad \frac{d\vec{V}}{dt} = \frac{e}{m}\left(\vec{E} + \frac{\vec{V} \times \vec{B}}{c}\right) \tag{3.1}$$

where $\vec{X}$ is the particle position and $\vec{V}$ its velocity. The equations obviously separate into motion parallel and perpendicular to $\vec{B}$. The constant $\vec{E}$ & $\vec{B}$ fields the equations are trivally solved to give

$$\begin{aligned} X_{\parallel} &= X_{\parallel 0} + V_{\parallel 0}\, t + \frac{q}{2m} E_{\parallel} t^2 \\ V_{\parallel} &= V_{\parallel 0} + \frac{e}{m} E_{\parallel}\, t \\ \vec{X}_{\perp} &= \vec{V}_D\, t + \vec{\rho} \end{aligned} \tag{3.2}$$

where the electric drift velocity is $V_D = \frac{\vec{E} \times \vec{B}}{B^2} c$ and $\vec{\rho}$ is a circular motion in the drift frame with frequency, $\Omega = eB/mc$ and Larmor radius $\rho = v_{\perp}/\Omega$. It is important to notice that the drift-velocity is the same for ions and electrons, being independent of mass and charge, but that the cyclotron frequencies and gyroradii are not. The electric field only <u>accelerates</u> particle parallel to $\vec{B}$ and because electrons respond so quickly it is difficult to maintain a constant $E_{\parallel}$, except in a potential well generated by a collection of (magnetically trapped) ions. We assume $E_{\parallel} = 0$ for the moment.

In a real device we need to understand the motion of particles in electromagnetic fields which vary on time scale t and space scales $L_\perp$, $L_\parallel$. In the case where $\Omega\tau$, $\rho/L_\perp$, $\rho/L_\parallel$ are all $O(1)$ only a high degree of symmetry will save you from needing a computer, but otherwise there are various forms of perturbation theory which give approximate solutions. The most useful cases will be discussed.

1) Small Larmor Radius, Slow time Scales.

The case with $\Omega\tau >> 1$, $\rho/L_\perp$, $\rho/L_\parallel << 1$ is of the most interest. Since the gyrofrequency is large it seems appropriate to average out htis rapid motion and develop equations for the mean drift of the particle. As the particle moves its local gyrofrequency will change; consider the Taylor series expansion of the function

$$x = x_0 \cos (\Omega_0 + \varepsilon\Omega)t \tag{3.3}$$

$$= x_0 \cos \Omega_0 t + x_0 t\, \varepsilon\Omega \sin \Omega_0 t - \frac{x_0}{2} (t\varepsilon\Omega)^2 \cos \Omega_0 t + \ldots$$

The successive terms are not purely oscillatory, but are 'secular' with coefficients $O(t^n)$. The radius of convergence of the series is $O(\Omega_0^{-1})$ and so simple Taylor expansion of the equation of motion is of little value and we prefer the method of averaging which, although it is only asymptotic, has a range $O\left(\frac{1}{\varepsilon\Omega_0}\right)$ or better. The method is required in many applications and so is worth giving here in detail.

The original equations of motion must be normalized and transformed $(\vec{x},\vec{v}\to\vec{y},\nu)$ to display the phase angle ν of the gyromotion as follows (McNamara and Whiteman, 1967).

$$\vec{Y}_t = \varepsilon\, \vec{g}\, (\vec{Y},\nu) \tag{3.4}$$

$$\nu_t = 1 + \varepsilon f(\vec{Y},\nu)$$

where $\vec{g}$ and f are periodic in ν, period τ_0, and ε is the small expansion

The equations (3.8, 3.9) are sufficient to generate a power series expansion for the transformation $\vec{Z}$ and the averaged driving terms $\vec{H}$. The result is conveniently written down in terms of the integrating and averaging operators ($\sim$,-) defined as follows: for any function f, periodic in ν,

$$\begin{aligned} \tilde{f} &= \int_0^{\nu} (f - \bar{f})d\nu' \\ \bar{f} &= \frac{1}{\tau_0}\int_0^{\tau_0} f d\nu' \end{aligned} \tag{3.10}$$

Notice that $\tilde{f}$ contains a constant of integration or initial condition and so $\bar{\bar{f}} = 0$ but $\tilde{\tilde{f}} \neq 0$. Without further ado, we write the transformation to $O(\epsilon^3)$ as

$$\begin{aligned} \vec{Z} &= \vec{Y} - \epsilon\tilde{\vec{G}} + \epsilon^2\left(\widetilde{\tilde{\vec{G}}\cdot\tilde{\tilde{\vec{G}}}_Y} - \tilde{\tilde{\vec{G}}}\cdot\bar{\vec{G}}_Y\right) + O(\epsilon^3) \\ \vec{Y} &= \vec{Z} + \epsilon\tilde{\vec{G}} + \epsilon^2\left(\widetilde{\tilde{\vec{G}}\cdot\tilde{\vec{G}}_Z} - \bar{\vec{G}}\cdot\tilde{\tilde{\vec{G}}}_Z\right) + O(\epsilon^3) \end{aligned} \tag{3.11}$$

The average coordinates $\vec{Z}$ have the equation of motion

$$\vec{Z}_t = \vec{a} + \epsilon\bar{\vec{G}}(\vec{Z}) - \epsilon^2\left(\overline{\tilde{\vec{G}}\cdot\tilde{\vec{G}}_Z} - \bar{\tilde{\vec{G}}}\cdot\bar{\vec{G}}_Z\right) + O(\epsilon^3) \tag{3.12}$$

where $\vec{a} = (1,0,0,\ldots,0)$. Notice that the phase $\dot{\phi}$ does not appear on the right of this equation, as desired. There are many descriptions of the method of averaging in the text books but equations (3.11, 3.12) are the answer for the plasma physicist. In celestial mechanics one is usually interested in a high order of accuracy and so requires many orders of the expansion. The best method is due to Deprit (1969) and is well described in Nayfeh's book (1973). The method uses a generating function or Lie transform, $\vec{W}(\vec{Y})$, which allows the manipulations to be computerized on an algebraic manipulator. This generating function approach also allows any function of the old variables to be expanded directly in the new variables.

We observe that the original problem has merely been transformed to a simpler one which still must be solved, equations (3.12). As a final answer, slower period oscillation. The same technique can be used to average over this oscillation and reduce the system still further.

3.2 High Frequency Fields

In the case when $\Omega_\tau \ll 1$, $\tau v/L \ll 1$, the equations of motion can be averaged over the high-frequency field variation. In the non-relativistic case we get

$$m\,\frac{d\vec{V}}{dt} = e\bar{E} + \frac{e}{c}\,(V \times \vec{B}) - \frac{e^2}{2m\omega^2}\,\nabla\overline{\tilde{E}^2} \qquad (3.16)$$

The high frequency part of the field appears as a potential $u = (e^2/2m\omega^2)\overline{\tilde{E}^2}$, independent of the sign of the charge. This additional force is of prime importance in laser fusion. When $\bar{E}$, $\bar{B}$ vary slowly on the scale of the larmor period the method of averaging can be applied, as before.

4. Adiabatic Invariants

The six equations of motion have six constants of the motion, namely the initial conditions on the motion. These are in general useless for making further deductions and we seek a better choice of constants in systems with sufficient symmetry. A typical example for a charged particle in a time independent field is the total energy or the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi \qquad (4.1)$$

If the fields $\vec{B} = \nabla \times \vec{A}$, $\vec{E} = -\nabla\phi$, are independent of a coordinate, θ, then the corresponding cauonical momentum, P_θ, is a constant of the motion. Such constants confine a particle to a surface in phase space which, if we are lucky or chose the configuration carefully, will confine where G_n is independent of q_1. We finally require that J_n be periodic in q_1 and so the average of the Poisson bracket must be made to vanish by choice of J_0 and the integration constants G_n:

$$\left[\overline{J_{n-1}, \Omega}\right] = 0 \tag{4.7}$$

The first equation is

$$\left[\overline{J_0, \Omega}\right] = 0 = \left[J_0, \bar\Omega\right] \tag{4.8}$$

since J_0 is independent of q_1. The obvious non-trivial solution is

$$J_0 = J_0\,(\bar\Omega). \tag{4.9}$$

The series can be developed in terms of the Poisson bracket operator, the averaging operator, and the indefinite integrator

$$\hat J = \int (J - \bar J)\, dq_1 \tag{4.10}$$

and the general answer, correct to $O(\varepsilon^2)$ is

$$\begin{aligned} J = \bar\Omega &+ \varepsilon\,[\bar\Omega, \hat\Omega] + \frac{\varepsilon}{2}\,\overline{[\hat\Omega, \Omega]} \\ &+ \varepsilon^2\left[[\bar\Omega, \hat\Omega] + \frac{1}{2}\,\widehat{\overline{[\hat\Omega, \Omega]}, \Omega}\right] \\ &+ \frac{\varepsilon^2}{3}\left[\overline{\hat\Omega\,[\hat\Omega, \Omega]}\right] + \frac{2\varepsilon^2}{3}\left[\overline{\hat\Omega, [\hat\Omega, \bar\Omega\,]}\right] + O(\varepsilon^3) \end{aligned} \tag{4.11}$$

The most general Hamiltonian for which we have developed such an adiabatic invariant is of the form

$$H = \psi\left(\varepsilon q_2, \cdots, \varepsilon q_N, P_1, \varepsilon P_2, \ldots, \varepsilon P_N, \varepsilon t\right) \tag{4.12}$$

$$+ \varepsilon\Omega\,(q_i, P_i, \varepsilon t)$$

In terms of the rotation frequency $\lambda = \partial\psi/\partial P_1$ and the "slow" bracket { }

$\vec{\rho} = -\hat{b}\cdot\nabla\hat{b}$, the field line curvature.

Notice that μ, contains $V_\perp$ and the oscillates on the cyclotron period. When the particles are trapped in a magnetic mirror field to a particular region of a field line we have to introduce the sign, $\sigma = \pm 1$, of the parallel velocity. The bounce motion, at frequency ω_b, can be averaged to yield a second invariant provided $\Omega \gg \omega_b \gg |W_D/L|$:

$$J = \oint \left(2\left(\epsilon - \left(\mu_0 + \frac{m}{e}\mu_1\right)B\right)\right)^{\frac{1}{2}} ds - \sigma \int_{S_0}^{S} \frac{dS}{q} V_D\cdot\nabla J_0 + 0\left(\frac{r_L}{L}\right)^2 \qquad (4.16)$$

where $J_0 = \oint 2(\epsilon - \mu_0 B)^{\frac{1}{2}} dS$

Finally, if the fields are allowed to change very slowly in time over a period much longer than the drift of a particle around a J surface then the energy ϵ is no longer a constant of motion on this time scale. The total flux Φ through a drift surface, J = const, is another adiabatic invariant:

$$\Phi = \oint_{J = \text{const}} \alpha d\beta \qquad (4.17)$$

where (α,β) are the field line coordinates, $\vec{B} = \nabla\alpha \times \nabla\beta$.

For a plasma to be in equilibrium in a given static magnetic field the plasma distribution function must be a function of the constants of motion, $f = f(\epsilon,\mu,J)$. This statement will be developed better in the talk on mirror machines.

5. Analogy with Magnetic Field Structures

As an aside to the main business of particle motions the structure of magnetic fields can be analyzed in the same fashion. Compare the general invariants for a divergence free magnetic field and a Hamiltonian system, namely the flux Φ through an arbitrary curve C which always passes through the same set of field lines and the action integral J around an arbitrary loop

The solution can be written down easily from eq. (4.11):

$$\psi = \overline{b_x^{\mp}} - \overline{b_y^{*}} + \epsilon \; \overline{b_x \hat{b}_{y1}} + \epsilon \left[\overline{b_x^{+}} - \overline{b_y^{*}} \, , \, \hat{b}_x^{+} - \hat{b}_y^{*} \right] + O(\epsilon^2) \tag{5.8}$$

In the stellarator configuration it is usually assumed that $\overline{b_x^{\perp}}, \overline{b_y^{*}}$ are $O(\epsilon)$ and so the form of the surfaces is determined by

$$\psi = \epsilon \, \overline{b_x \hat{b}_{y1}} \, . \tag{5.9}$$

This work is one illustration of how to take a conservative ($\nabla \cdot \vec{B} = 0$) system and express it in Hamiltonian form and shows how the discussion of particle orbits relates to magnetic surfaces. Typical magnetic surfaces in an $L = 3$ stellarator are shown in Fig. 1. (A. Gibson 1967).

6. Resonant Effects on Adiabatic Invariants.

The theory of invariants so far described shows how to average over a single frequency. In systems where there is more than one fundamental frequency or where the fundamental varies in phase space it is possible for beats between the various frequencies to produce a slow variation. Terms like $\cos(n\omega_1 - m\omega_2)q$, arise in the series expansions and when integrated have a denominator $(n\omega_1 - m\omega_2)$ which could be very small for large values of n,m. The series can only be shown to be asymptotic and one simply has to stop the expansion when a small denominator arises. If this happens in the second or third term then the whole procedure must be modified.

A nice example (Taylor and Laing 1976) is of a charged particle in a uniform magnetic field interacting with an electrostatic plasma wave. The Hamiltonian is

$$H(\vec{r},\vec{p}) = (\vec{p} - m\Omega x\hat{y})^2/2m + e\Phi_0 \sin(kz + k_\perp x) \tag{6.1}$$

surfaces have broken up, leaving only islands around certain fixed points of the phase plane. Jaeger and Lichtenberg have examined a number of simpler examples and distinguish two possible ways in which the resonant surfaces can break up. In the case of an exact resonance a 2D oscillator problem can be reduced to the averaged Hamiltonian

$$\bar{H} = K + \epsilon \bar{\Omega} (Q_2, K, P_2) + O(\epsilon^2) \tag{6.7}$$

where $K = \tau H - \epsilon\tau J$ is the canonical invariant conjugate to Q_1 in sec. (4). The remaining motion, in Q_2, P_2 may have an elliptic fixed point $(\bar{Q}_2, \bar{P}_2)$, where

$$\frac{\partial \bar{H}}{\partial Q_2} = 0 \quad \frac{\partial \bar{H}}{\partial P_2} = 0 \tag{6.8}$$

Expanding about this point we get the Hamiltonian for the local motion $\delta Q_2 = Q_2 - \bar{Q}_2$, $\delta P_2 = P_2 - \bar{P}_2$:

$$\bar{H} = K + \epsilon \bar{\Omega} (\bar{Q}_2, K, \bar{P}_2) + \epsilon \frac{\delta Q_2}{2} \left(\frac{\partial^2 \bar{\Omega}}{\partial Q_2^2}\right) + \epsilon \frac{\delta P_2^2}{2} \left(\frac{\partial^2 \bar{\Omega}}{\partial P_2^2}\right) + O(\delta^3) \tag{6.9}$$

The frequency of these oscillations is clearly $O(\epsilon)$ and the ratio of the semiaxes of the orbits in the $(\delta Q_2, \delta P_2)$ phase plane is

$$R = \frac{(\delta P_2)_{MAX}}{(\delta Q_2)_{MAX}} = O\left(\left(\frac{\partial^2 \bar{\Omega}}{\partial Q_2^2}\right) \Big/ \left(\frac{\partial^2 \bar{\Omega}}{\partial P_2^2}\right)\right)^{1/2} \tag{6.10}$$

If a high harmonic of these oscillations resonates with the primary oscillations in the (K, Q_1) phase space then the invariant is altered just as described above for the magnetized particle in a wave. That example was more complicated in that the resonance between the ϕ and z oscillations depended on p_z. The best we can do with the invariant is to write the Hamiltonian as

$$\bar{H} = \psi (I, P_2) + \epsilon \bar{\Omega} (I, P_2, Q_2) + O(\epsilon^2) \tag{6.11}$$

The expansion about an elliptic point in this case gives of the series is important. There are cases of simple dynamical systems where an exact constant can be found which, on expansion in the appropriate small parameter gives the adiabatic series. In general, the best that can be done is to show that the series is asymptotically convergent: The general invariant J, of eq. (4.11), summed to n terms can be shown to vary like

$$\frac{dJ^{[n]}}{dt} = O\left(\varepsilon^{n+1}\right) \tag{7.1}$$

A key assumption in the whole development was that J could be expanded in a power series in ε, which a priori eliminates small non-expandable terms like $ae^{-b/\varepsilon}$. The magnetic moment, μ, of a particle displays just such jumps at each bounce of the particle in a mirror machine (Cohen et. al., Hastie et. al.). This is easily deduced by examining the change in μ_0 over one bounce. The exact equations of motion give

$$\frac{d\mu_0}{dt} = -\frac{v_\perp}{2B}\left(v_\perp^2 + 2v_\parallel^2\right)\rho_\perp \cos\psi + \frac{v_\parallel^2}{B} v_\perp \rho_J \cos\psi_J$$
$$-\frac{v_\parallel}{B}\left[\mu_0 \frac{\partial B}{\partial s} + \vec{v}_\perp \cdot (\vec{v}_\perp \cdot \nabla) \cdot \hat{b}\right] \tag{7.2}$$

where

$$\cos\psi = (v_\perp \cdot \nabla B)/(v_\perp|\nabla B|), \quad \cos\psi_J = \vec{v}\cdot\vec{\rho}_J/(v_\perp \rho_J)$$

$$\vec{\rho}_J = \frac{\vec{B}\times(\nabla\times\vec{B})}{B^2} \qquad \rho_\perp = |\nabla_\perp B|/B$$

$$\vec{v} = v_\perp \cos\bar{\psi}\,\hat{e}_1 - v_\perp \sin\bar{\psi}\,\hat{e}_2 + v_\parallel \hat{b}$$

This equation is integrated along a field line, the zeroth order motion of the particle, to give the change in μ_0:

$$\Delta\mu_0 = \sqrt{2\mu_0}\; v^2 \,\mathrm{Re} \oint \frac{ds}{v_\parallel} \frac{\rho_\perp e^{i\psi} + \rho_J}{\sqrt{B}} \; e^{i\psi_J} \tag{7.3}$$

A - Atomic mass τ - bounce time

M - Mass z - charge

W - energy

Notice that these results arise from a non-resonant coupling between the bounce motion and the cyclotron motion and account for the stochastic motions in phase space when the adiabatic invariant has broken down. One remaining question is whether or not these exponentially small jumps destroy the invariance of μ over a long time scale even in the adiabatic region.

8. Superadiabaticity

This has been investigated by Aamodt and by Rosenbluth for the case of mirror trapped particles in the presence of electrostatic fluctuations near a harmonic of the cyclotron frequency which produces similar jumps in μ. The key point is that jumps in μ_0 are periodic in ψ_0. Let ψ_n be the phase on the nth bounce and μ_n the magnetic moment on the prior to the nth scattering, then (7.6) can be rewritten as

$$\mu_{n+1} = \mu_n + \alpha \sin \psi_n \tag{8.1}$$

The particle makes many gyrations between bounces and we need a simple model to describe ψ_{n+1} in terms of ψ_n and μ. Following Rosenbluth, let us consider a simple quadratic variation in field strength so that the cyclotron frequency is

$$\Omega = \Omega_0\left(1 + s^2/L^2\right) \tag{8.2}$$

Constancy of the total energy gives

$$\frac{1}{2} V_\parallel^2 = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 + \mu B_0 \frac{s^2}{L^2} \tag{8.3}$$

When this condition is met the particle orbits do not diffuse in velocity space due to the non-adiabatic jumps in μ, and the orbits are called "super-adiabatic". Numerical calculations by Cohen et. al. show that particle orbits in typical mirror fields are indeed superadiabatic up to about twice the energy at which the jumps could compete with coulomb scatteri ıg in a fusion plasma, eqn. (7.8). The adiabatic invariants of a charged particle are indeed approximations to gooû constants of the motion.

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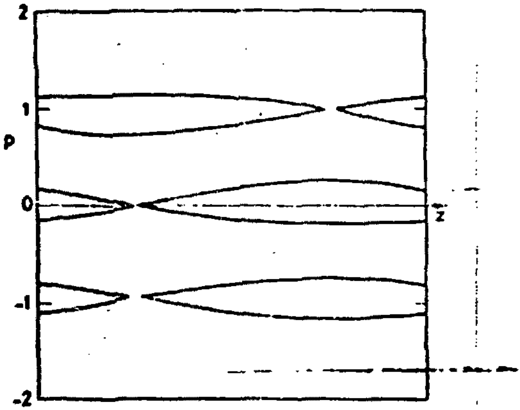

FIG. 2 Surface of section plot of $I_0 + \epsilon I_1$, $\epsilon = .025$.
Taylor and Laing (1976).

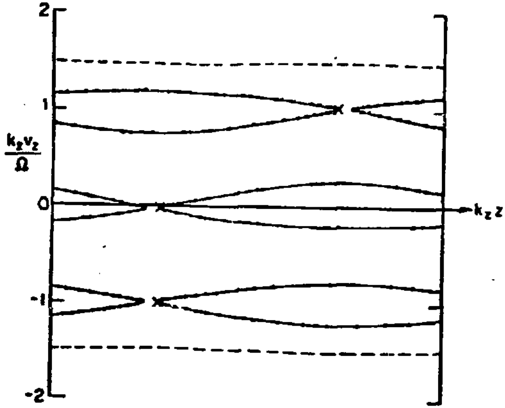

FIG. 4 Numerical Orbit computations at $\varepsilon = .025$ by Smith and Kaufman (1976).

2. Ionization of Neutral Atoms and Molecules

It seems likely that the primary heating and fueling of fusion reactors will be by hot neutral injection. The neutral atom or molecule is therefore a starting point for the discussion of the physics of single particle behavior. A neutral beam injector works by accelerating ions, say D^+, out of a low temperature plasma, and then neutralizing the fast ions by charge exchange with some suitable gas, say D_2:

$$D_2 + D^+_{HOT} \rightarrow D_{HOT} + D_2^+ \qquad (2.1)$$

The hot neutral enters the fusion plasma and can be ionized by the ions or electrons of the plasma:

$$D^+ + D \rightarrow 2D^+ + e \qquad (2.2)$$

$$e + D \rightarrow D^+ + 2e \qquad (2.3)$$

or can charge exchange with a hot-ion of the plasma which will produce a hot neutral traveling in a different direction - probably out of the plasma. The measurement and calculation of the cross sections for these and similar reactions if an elaborate process and I do not wish to give a course on *Atomic Physics! (one has just been given at Trieste). However, these cross* sections are important to those who wish to compute the behavior of neutral injectors and the buildup and equilibrium of injected plasmas. Many of these cross sections for the interaction of atoms and molecules of hydrogen, deuterium, and tritium with themselves, electrons, and α-particles have been incorporated in a simple subroutine by C.A. Finan.

Many other such efforts exist to reduce the relevant data on cross sections, atomic energy

3. Motion of Charged Particles in Electromagnetic Fields

Charged particle motions are generally complicated and, in designing fusion devices, one tries to simplify the motions by use of symmetries or constants of the particle motions to provide confinement within the device. The equations of motion in an electric field $\vec{E}(\vec{x},t)$ and magnetic field $\vec{B}(X,t)$ are, in Gaussian units,

$$\frac{d\vec{X}}{dt} = \vec{V} \quad , \quad \frac{d\vec{V}}{dt} = \frac{e}{m}\left(\vec{E} + \frac{\vec{V} \times \vec{B}}{c}\right) \tag{3.1}$$

where $\vec{X}$ is the particle position and $\vec{V}$ its velocity. The equations obviously separate into motion parallel and perpendicular to $\vec{B}$. The constant $\vec{E}$ & $\vec{B}$ fields the equations are trivally solved to give

$$\begin{aligned} X_{\parallel} &= X_{\parallel 0} + V_{\parallel 0}\, t + \frac{q}{2m} E_{\parallel} t^2 \\ V_{\parallel} &= V_{\parallel 0} + \frac{e}{m} E_{\parallel}\, t \\ \vec{X}_{\perp} &= \vec{V}_D\, t + \vec{\rho} \end{aligned} \tag{3.2}$$

where the electric drift velocity is $V_D = \frac{\vec{E} \times \vec{B}}{B^2} c$ and $\vec{\rho}$ is a circular motion in the drift frame with frequency, $\Omega = eB/mc$ and Larmor radius $\rho = v_{\perp}/\Omega$. It is important to notice that the drift-velocity is the same for ions and electrons, being independent of mass and charge, but that the cyclotron frequencies and gyroradii are not. The electric field only accelerates particle parallel to $\vec{B}$ and because electrons respond so quickly it is difficult to maintain a constant $E_{\parallel}$, except in a potential well generated by a collection of (magnetically trapped) ions. We assume $E_{\parallel} = 0$ for the moment.

parameter, $O\left(\rho/L_\perp,\ \rho/L_\parallel,\ \Omega\tau^{-1}\right)$. We construct a transformation to new variables $(\vec{Z},\ \phi)$ with $\vec{Z}$ periodic in ν and ϕ being an angle variable:

$$\vec{Z}(\vec{Y},\nu) = Z(\vec{Y},\ \nu + \tau_0) \qquad (3.5)$$

$$\phi(\vec{Y},\ \nu + \tau_0) = \tau_0 + \phi(\vec{Y},\ \nu)$$

The equations for the drift variables $(\vec{Z},\ \phi)$ should not contain the angle variable which is to be averaged out:

$$\vec{Z}_t = \epsilon\vec{h}(\vec{Z}) \qquad (3.6)$$

$$\phi_t = 1 + \epsilon\omega(\vec{Z})$$

The original equations (3.4) and the transformation give

$$\vec{Z}_t = \epsilon\vec{g}\cdot\vec{Z}_{\vec{Y}} + (1 + \epsilon f)\ \vec{Z}_\nu \equiv \epsilon\vec{h} \qquad (3.7)$$

$$\phi_t = \epsilon\vec{g}\cdot\phi_{\vec{Y}} + (1 + \epsilon f)\ \phi_\nu \equiv 1 + \epsilon\omega$$

which can be integrated over ν. We combine the equations into a single vector equation by setting $\vec{Z} = (\phi,\ \vec{Z})$, $\vec{H} = (\omega,\vec{h})$, $\vec{Y} = (\nu,\vec{Y})$ and $G(\vec{Y}) = (f,\vec{g})$ and the integration, with boundary condition $\vec{Z}(0) = \vec{Y}$, gives

$$\vec{Z} = \vec{Y} + \epsilon\int_0^\nu d\nu\ \left(\vec{H}\ (\vec{Z}) - \vec{G}\cdot\vec{Z}_{\vec{Y}}\right) \qquad (3.8)$$

The condition that $\vec{Z}$ have no secular terms in ν is that the average of the integrand vanish:

$$\int_0^{\tau_0} d\nu\ (\vec{H} - \vec{G}\cdot\vec{Z}_{\vec{Y}}) = 0 \qquad (3.9)$$

which will appear many times at this college, we can write down the result of applying this method to eliminating the gyrorotation from the equations of motion of a charged particle (3.1). These are the well known drift-equations: (Morozov and Soloviev, 1966).

$$\frac{d\vec{X}}{dt} = v_{\parallel}\frac{\vec{B}}{B} + \frac{c}{B^2}(\vec{E}\times\vec{B}) + \frac{mcv_{\parallel}^2}{eB^4}\vec{B}\times(\vec{B}\cdot\nabla)\cdot\vec{B} + \frac{mcv_{\perp}^2}{2eB^3}\vec{B}\times\nabla B \tag{3.13}$$

$$\frac{d\varepsilon}{dt} = e\vec{E}\cdot\frac{dX}{dt} + \frac{mv_{\perp}^2}{2B}\frac{\partial B}{\partial t}$$

$$\frac{d}{dt}\left(\frac{cm^2v_{\perp}^2}{m_0^2Be}\right) = 0 \equiv \frac{d\mu_0}{dt}$$

$$\frac{d\phi}{dt} = \Omega$$

where the energy of the realivistic particle is $\varepsilon = m_0c^3/\left(c^2 - v_{\parallel}^2 + v_{\perp}^2\right)^{1/2}$ or, for a non-relativistic particle is $\frac{m}{2}(v_{\parallel}^2 + v_{\perp}^2)$. We observe that, to this order, the perpendicular velocity is determined by the constant of the motion, the adiabatic invariant μ_0. When $\nabla\times\hat{B} = 0$ the magnetic drifts are of the same form and we get

$$\frac{d\vec{X}}{dt} = v_{\parallel}\frac{\vec{B}}{B} + \frac{c}{B^2}(\vec{E}\times\vec{B}) + \frac{mc^2}{2eB^3}(\vec{B}\times\nabla B)(2v_{\parallel}^2 + v_{\perp}^2) \tag{3.14}$$

One essential assumption in the derivation was that $E \ll v\,B/c$. If we allow for a large drift-velocity, $\vec{v}_E = \frac{\vec{E}\times\vec{B}}{B^2}c$ the equations are modified to

$$\frac{d\vec{X}}{dt} = \vec{U} - \frac{mc}{eB^2}\left[\frac{\partial\vec{U}}{\partial t} + (\vec{U}\cdot\nabla)\cdot\vec{U}\right]\times\vec{B} + \frac{mcv_{\perp}^2}{2eB^3}\vec{B}\times\nabla B \tag{3.15}$$

$$\varepsilon = \left(\frac{m}{2}\; v_{\parallel}^2 + v_{\perp}^2 + v_E^2\right), \qquad \vec{U} = v_{\parallel}\frac{\vec{B}}{B} + \vec{v}_E$$

The second term in (3.15) is the drift due to the inertial effect of the large electric drift. These drift equations are very useful in determining the dynamics of a plasma on time scales long compared with the cyclotron period. In some cases the drift equations themselves will describe a still the particle in configuration space. One method for finding constants in less symmetric situations is to transform the Hamiltonian to momentum coordinates which display the larmor angle. We could then seek a canonical transformation, as a power series in r_L/L, which would make the Hamiltonian independent of the new phase angle and hence the corresponding momentum would be a constant. Unfortunately, it would be expressed in terms of the averaged variables and we would then have to find its expansion as a series in the original phase space coordinates.

We will demonstrate a different formulation which is of general usefulness. Consider systems in which the particles execute closed orbits in the unperturbed system so the Hamiltonian can be reduced to

$$H = P_1 + \epsilon\Omega(q_i, P_i) \tag{4.2}$$

where Ω is (almost) periodic in the angle coordinate q_1. Then we look for a constant of the motion J by solving the linear, partial differential equation

$$\frac{dJ}{dt} \equiv [J,H] = 0 \tag{4.3}$$

where [J,H] is the Poisson bracket

$$[J,H] \equiv \sum_i \left(\frac{\partial J}{\partial P_i} \frac{\partial H}{\partial q_i} - \frac{\partial J}{\partial q_i} \frac{\partial H}{\partial P_i} \right) \tag{4.4}$$

J is expanded as a power series, $\sum_0^\infty \epsilon^u J_u$, to give the recursion

$$\frac{\partial J_0}{\partial q_1} = 0 , \qquad \frac{\partial J_n}{\partial q_1} = \left[J_{n-1}, \Omega\right] \tag{4.5}$$

The n^{th} equation is easily integrated,

$$J_n = \int \left[J_{n-1}, \Omega\right] dq_1 + G_n \tag{4.6}$$

defined as

$$[\psi,f] = \lambda \frac{\partial f}{\partial q_1} + \varepsilon \ (\psi,f) \tag{4.13}$$

the result for a general oscillatory system is

$$J = P_1 + \varepsilon\left(\frac{\Omega - \overline{\Omega}}{\lambda}\right) + \varepsilon^2\left(\frac{1}{\lambda}\left\{\frac{\hat{\Omega}}{\lambda},\psi\right\} + \left[\frac{1}{2\lambda^2},\frac{\hat{\Omega}^2}{2}\right] - \frac{1}{\lambda}\left[\frac{\overline{\Omega}}{\lambda},\hat{\Omega}\right] - \frac{1}{\lambda}\frac{\partial}{\partial \varepsilon t}\left(\frac{\hat{\Omega}}{\lambda}\right) + \frac{1}{2}\overline{\left[\frac{\Omega}{\lambda},\frac{\hat{\Omega}}{\lambda}\right]}\right) + O(\varepsilon^3) \tag{4.14}$$

As far as plasma theory is concerned we can regard the calculation of adiabatic invariants as being solved. However, the Hamiltonian formulation is most inconvenient since H is a function of the potentials (A,ϕ) and not the fields (B,E). The above method really only required the equation $[J,H] = 0$ to be expressed in coordinates which display the phase angle over which we average. Haas, Hastie, and Taylor have applied the method to the Vlasov equation to generate the magnetic moment μ, the longitudinal invariant J, and the flux invarient Φ as given below. The algebra involved was formidable and the results are worth some comments.

A charged particle in a time independent electromagnetic field will have as constants of the motion the energy ε and the canonical moments corresponding to any symmetrics of the configuration. If there are no symmetries then, in a strong magnetic field such that $r_L/L \ll 1$, the magnetic moment will be an adiabatic constant:

$$\mu = \mu_0 + \frac{m}{e}\mu_1 + O\left(\frac{r_L}{L}\right)^2 \tag{4.15}$$

where $\mu_0 = v_\perp^2/2B$

$$\mu_1 = -\frac{1}{B}\left[\vec{V}_\perp\cdot\vec{W}_D + \frac{\vec{V}\cdot b}{4}\left\{\vec{V}_\perp\cdot(\vec{a}\cdot\nabla)\vec{b} + \vec{a}\cdot(\vec{V}_\perp\cdot\nabla)\vec{b} + 4\mu_0\vec{b}\cdot\nabla\times\vec{b}\right\}\right]$$

and $W_D = \frac{\vec{B}}{B^2} \times (V_\parallel^2\,\vec{\rho} + \mu\nabla B)$, $\quad \vec{a} = \frac{\vec{V}\times\vec{B}}{B^2}$

in phase space which always passes through the same trajectories:

$$\Phi = \iint_C B_z \, dxdy, \qquad J = \oint p dq \tag{5.1}$$

This suggests that a magnetic field might be described in canonical coordinates

$$p \equiv \int^y B_z dy = B_z^{+}, \quad q \equiv x, \quad t \equiv z \tag{5.2}$$

The Hamiltonian may be found from the equations of motion

$$-\frac{\partial H}{\partial q} = \frac{dp}{dt}, \quad \frac{\partial H}{\partial p} = \frac{dq}{dt} \equiv \frac{dx}{dz} = \frac{B_x}{B_z} \tag{5.3}$$

and the constraint $\nabla \cdot \vec{B} = 0$. It turns out that we need to separate the field component $B_y = B_{y1}(x,y,z) + B_{y2}(x,z)$ to correctly choose the constants of integration when solving for H :

$$H = \int^y B_x dy - \int^x B_{y2} dx \equiv B_x^{+} - B_y^{*} \tag{6.4}$$

As a simple example, we can now apply the Hamiltonian formalizm to a stellarator field which we write as

$$\vec{B} = B_0\hat{z} + \epsilon\, \vec{b}_1(x,y,z,\epsilon) \tag{5.5}$$

where the field is principally in the z direction ($\hat{z}$ - unit vector) and $\vec{b}_1$ is periodic in z and may be further expandable in ϵ. The momentum and Hamiltonian are

$$p = B_0 y + \epsilon\, b_z^{+}, \; H = \epsilon b_x^{+} - \epsilon b_y^{*} \equiv \epsilon h \tag{5.6}$$

We seek an adiabatic invariant ψ to describe the magnetic surfaces by solving the Hamiltonian form of $\vec{B}.\nabla\psi = 0$, namely,

$$\frac{d\psi}{dt} = 0 = \frac{\partial\psi}{\partial t} + \epsilon\,[h,\psi] \tag{5.7}$$

where $\Omega = eB/mc$ and ϕ_0 is the wave amplitude. This is first transformed to the action-angle coordinates of the gyromotion, $P_\phi = mv_\perp^2/2\Omega$, with gyroradius $\rho = (2P_\phi/m\Omega)^{1/2}$:

$$H = P_z^2/2m + \Omega P_\phi + e\Phi_0 \sin (kz - k_\perp \rho \sin \phi) \tag{6.2}$$

In the case of propagation at 45° ($k = k_\perp$), the Hamiltonian may be non-dimensionalized and, using a Bessel function identity, becomes

$$H = P_\phi + \frac{p^2}{2} + \varepsilon \sum J_L (\rho) \sin (z - \ell\phi) \equiv H_0 + \varepsilon H_1 \tag{6.3}$$

The recurrence relations in (4.5) for an invariant I become

$$\frac{\partial I_0}{\partial \phi} + p \frac{\partial I_0}{\partial z} = 0, \quad \frac{\partial I_n}{\partial \phi} + p \frac{\partial I_n}{\partial z} = \left[H_1, I_{n-1}\right] \tag{6.4}$$

observe that the zeroth order orbit depends on p!

The solution to $O(\varepsilon)$ of (6.4) is

$$I_0 + \varepsilon I_1 = I_0(p) + \varepsilon \frac{dI_0}{dp} \sum J_L (\rho) \quad \frac{\sin(z - \ell\phi)}{p - L} \tag{6.5}$$

The expansion clearly fails at every integral resonance $p = \ell$ unless I_0 is chosen to vanish in the same way at each resonance. An appropriate choice is $I_0 = \cos (\pi p)/\pi$:

$$I_0 + \varepsilon I_1 = \pi^{-1} \cos (\pi p) - \varepsilon \sin (\pi p) \sum J_L \frac{\sin(z - L\phi)}{(P - L)} \tag{6.6}$$

This invariant is shown in Figs. (2, 3) in the plane $\phi = \pi$ for two values of ε and $\rho = \left(1.48^2 - p^2\right)^{1/2}$. We observe that resonances at $p = 0, 1$ overlap strongly in the second case.

These curves can now be compared with the numerical orbit calculations of Smith and Kaufman, Figs. (4,5), at the same parameter values. The orbits are plotted as they intersect the plane $\phi = \pi$ and, when the points lie on a smooth curve it is clear that the invariant is a good one. In Fig. 5 the

$$\bar{H} \approx \psi + \epsilon \bar{\Omega} + \frac{\delta P_2^2}{2} \left(\frac{\partial^2 \psi}{\partial P_2^2} + \epsilon \frac{\partial^2 \bar{\Omega}}{\partial P_2^2} \right) + \epsilon \frac{\delta Q_2^2}{2} \frac{\partial^2 \bar{\Omega}}{\partial Q_2^2} + O(\delta^3) \qquad (6.12)$$

The frequency of the drift in the $P_2 Q_2$ plane is now $O(\sqrt{\epsilon})$ and R is also $O(\sqrt{\epsilon})$. This resonates more readily with the fundamental and it is the overlap of these secondary islands, in either case, which leads to the stochastic behavior. The criterion used by Smith and Kaufman, based on overlap of the primary resonances is not accurate and their computations clearly show a secondary chain of five islands around one of the fixed points. The start of the required transformations must be done with the usual generating function approach:

$$S = S(P_{old}, Q) = - P_\phi Q_\theta - \frac{Q_x}{\pi} \cos \pi P \qquad (6.13)$$

so that

$$H = P_\theta + \frac{1}{2} \left(\cos^{-1} \pi P_x \right)^2 + \epsilon \sum J_L(\rho) \sin \left(Q_x \sqrt{1 - \pi^2 P_x^2} - p\phi \right)$$

which makes $P_x = \frac{1}{\pi} \cos \pi P$ the leading order invariant in the new coordinates. I have not carried out the rest of the analysis of this case, but this shows how to bring together the elements of the modern theory. Jaeger et al. have applied the theory to electron cyclotron resonance heating in mirror machines. They show that, as the electron energy increases, the high order (5th) resonance of the bounce motion with the cyclotron heating breaks up the invariant surfaces and places an upper limit on the attainable electron energy.

7. Jumps in Adiabatic Invariants

The question arises as to whether the adiabatic invariant series are approximations to some true constant or whether they are merely approximate constants. In fusion plasmas we certainly want to contain the particles much longer than a few hundred cyclotron periods and the question of the convergence

The phases ψ, ψ_j are rotating rapidly at the gyrofrequency and the integral *is close to zero. A more careful analysis is done by deforming the path* of integration S into the complex plane to pick up the residues around the zeroes of B. The details vary for each plasma configuration and so we will display the results for a finite $\beta, \rho(B)$ equilibrium in a mirror machine when it can be shown that

$$\nabla \times y(B)\,\vec{B} = 0, \qquad \rho_J = \rho_\perp \frac{B}{y}\frac{\partial y}{\partial B} \tag{7.4}$$

The field can then be expanded about the j^{th} zero in the complex S plane in the form

$$B = B_j\,\varepsilon^{\nu}\,(\psi - \psi_j)^{\nu}\ , \qquad \varepsilon = \frac{V}{\Omega L}\ ,\quad L = \left(2B/\frac{\partial^2 B}{\partial S^2}\right)^{1/2} \tag{7.5}$$

and the general result is

$$\frac{\Delta\mu_o}{\mu_o} = \frac{4\pi}{y}\sum_j \frac{\nu\varepsilon^{-\nu/2}}{\Gamma\left(\frac{\nu}{2}+1\right)}\ \mathrm{Re}\left[\left(\frac{B_o}{B_j}\right)^{1/2}\left(1+\frac{By'}{y}\right)_j\ e^{i\left(\psi_o + \nu\frac{\pi}{4}\right)}\ e^{-K_j/\varepsilon}\right] \tag{7.6}$$

where

$$K_j = -i\int_{S=0}^{S_j}\frac{B}{LB_o}\frac{ds}{v_\parallel} + i\varepsilon\int_0^{S_j} ds\ \hat{e}_i\cdot\frac{\partial\hat{e}_2}{\partial s} \tag{7.7}$$

These small jumps in μ_o can lead to a diffusion in velocity space and rapid loss of containment for the most energetic particles. The maximum energy particle which can be contained in a mirror machine has

$$W_{max} = \frac{10^{-3}\,K^2\,B_{(}^{\,2}\,L^2\,Z^2}{M(1 - .036\,\ell n A_o)^2}\ \text{keV}. \tag{7.8}$$

where

$$A_o = \frac{M}{2}\left(\frac{50\ \text{keV}}{W}\right)\left[\frac{L}{170\ \text{cm}}\ \frac{.5v}{\langle v_\parallel\rangle_{bounce}}\ \frac{10^{-2}\text{secs}}{\tau}\right]^2\left[\frac{15(1-R)}{A}\left(\frac{v_\perp}{v}\right)_c\frac{1}{.83\langle\cos\psi\rangle_{rms}}\right]^4 \tag{7.9}$$

where $V_\parallel$ is the parallel velocity at $s = 0$. The change in $\delta\psi$ between bounces is then

$$\Delta\psi = \int\left(\Omega - \Omega_o\right)dt = 2\int_0^{x_{max}} \frac{\Omega_o\left(s^2/L^2\right)ds}{\left(V_\parallel^2 - 2\mu B_o s^2/L^2\right)^{1/2}} \approx \left(\frac{L}{\rho_i}\right)\frac{\pi V_\parallel^3}{\left(2\mu B_o\right)^{3/2}} \qquad (8.4)$$

Expressing μB_o in units of $V_\parallel^2/2$ the phase change between bounces is

$$\psi_{n+1} = \psi_n + \left(\frac{L}{\rho}\right)\frac{\pi}{\mu_{n+1}^{3/2}} \qquad (8.5)$$

There are clearly many fixed points of the mapping (8.1), (8.5) whenever $\mu_{n+1} = \mu_n = (L/\rho m)^{2/3}$, $\psi_{n+1} = \psi_n + m\pi$. Let us linearize the motion about one such point, $\psi_{n\varepsilon} = \psi_F + \delta\psi_n$, $\mu_u = \mu_F + \delta\mu_n$ to get

$$\delta\mu_{n+1} = \delta\mu_n + \alpha\delta\psi_n \qquad (8.6)$$

$$\delta\psi_{n+1} = \delta\psi_n - \frac{3}{2}\left(\frac{L}{\rho\mu_F^{5/2}}\right)\delta\mu_{n+1}$$

Eliminating $\delta\psi_n$ gives

$$\delta\mu_{n+1} - \left(2 - \frac{\alpha 3}{2}\left(\frac{L}{\rho\mu_F^{5/2}}\right)\right)\delta\mu_n + \delta\mu_{n-1} = 0 \qquad (8.7)$$

and we look for solutions of the form $\delta\mu_n \sim \lambda^n$: for stability $|2| \leq 1$ which gives

$$\alpha < \frac{2}{3}\left(\frac{L}{\rho}\right)^{2/3} m^{-5/3} \qquad (8.8)$$

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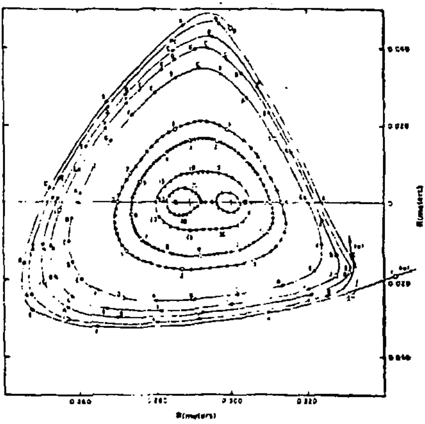

FIG. 1 Intersection of magnetic field lines with radial planes, at field period intervals, in a toroidal L = 3 Stellarator. There are 8 field periods around the torus and the numbers indicate revolutions about the major axis. The splitting of the axis arises from a small L = 1 field component due to the helical windings. The figure is taken from A. Gibson, (1967).

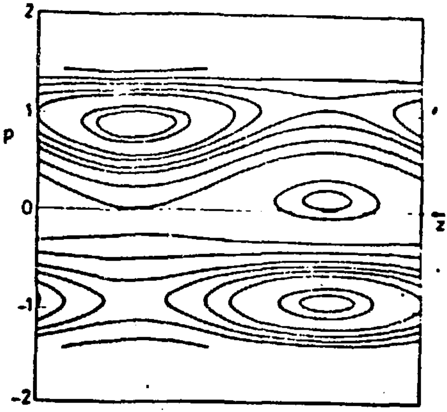

FIG. 3 Surface of section plot at ε = .1. Primary Resonances interact strongly. Taylor and Laing (1976).

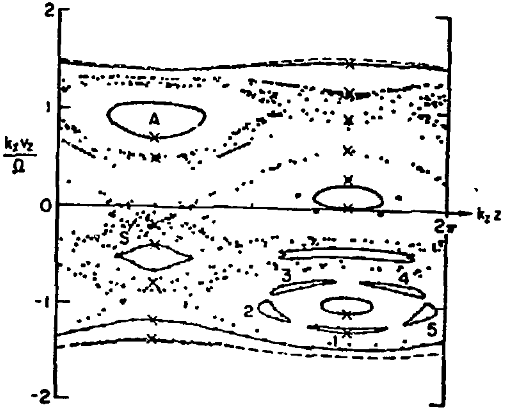

FIG. 5 Surface of section of orbits shows breakup of primary resonances and formation of $n_s = 5$ secondary resonances at $\epsilon = .1$. Smith and Kaufman (1976).